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# The period function for quadratic integrable systems with cubic orbits<sup>☆</sup>

Yulin Zhao

*Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel*  
*Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, PR China<sup>1</sup>*

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## Abstract

This paper is concerned with the monotonicity of the period function for families of quadratic systems with centers whose orbits lie on cubic planar curves. It is proved that each such system has a period function with at most one critical point.

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## 1. Introduction and main results

In this paper, we consider the monotonicity of period function of periodic trajectories for planar quadratic integrable system

$$\begin{aligned}\dot{x} &= P_2(x, y) = H_y(x, y)/M(x, y), \\ \dot{y} &= Q_2(x, y) = -H_x(x, y)/M(x, y),\end{aligned}\tag{1.1}$$

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*E-mail address:* [mcszyl@zsu.edu.cn](mailto:mcszyl@zsu.edu.cn).

<sup>1</sup> Current address.

where the dot denotes differentiation with respect to time  $t$ ,  $P_2(x, y)$  and  $Q_2(x, y)$  are quadratic polynomials of  $x, y$ ,  $H(x, y)$  is a first integral of system (1.1) with integrating factor  $M(x, y)$ . We assume that system (1.1) has at least one center. It is well known that there are at most two centers in a quadratic system and a center is surrounded by a continuous set of periodic trajectories. Each periodic orbit  $\Gamma_h$  is contained in a unique level set

$$\{(x, y) \mid H(x, y) = h\}$$

and its period equals to

$$T(h) = \oint_{\Gamma_h} dt = \oint_{\Gamma_h} \frac{dx}{P_2(x, y)} \quad (1.2)$$

for  $h \in \Sigma$ , where  $\Sigma$  is the maximal simple connected interval of existence of  $\Gamma_h$ . A center is called isochronous if  $T(h)$  does not depend on  $h$ .

Many authors have studied the monotonicity of period function for quadratic integrable system. For example, Chow and Sanders [6] show that system of the form  $x'' + f(x) = 0$ , with  $f(x)$  quadratic, necessarily have monotone period function. The quadratic isochronous system are completely classified by Loud [28]. Coppel and Gavrilov [15] proved that the period function of quadratic Hamiltonian system are monotone. Several authors [24, 37, 38, 46, 47] have shown that all Volterra–Lotka systems of the form

$$\dot{x} = x(a - by), \quad \dot{y} = y(cx - d)$$

have monotone period functions, too. Chicone and Dumortier proved in [3] that the period function for system

$$\dot{x} = -y + xy, \quad \dot{y} = x + 2y^2 - cx^2, \quad 0 < c < 2,$$

is not monotone for  $c \in (1.4, c^*)$ . Chicone and Jacobs [5] demonstrated some of the complexity displayed by the period functions for quadratic center by constructing a quadratic system with a center whose associated period function has at least two critical points. We studied the system  $Q_4: \dot{z} = -iz + 4z^2 + 2|z|^2 + (b + ic)\bar{z}^2$  and proved that its period function is monotone [50], where we use the terminology from [51] and  $z = x + iy$ ,  $|b + ic| = 2$ ,  $i^2 = -1$ . For more results on period function of plane systems, see [1, 6–8, 13, 14, 17, 32, 41, 42, 44] etc. and references therein.

Chicone has made the following conjecture [MR: 94h:58072]:

If a quadratic system has a center with a period function that is not monotonic, then, by an affine transformation and a constant rescaling of time, the system can be transformed to the Loud normal form

$$\dot{x} = -y + Bxy, \quad \dot{y} = x + Dx^2 + Fy^2.$$

Moreover, a system in Loud normal form has a center at the origin with a period function that has at most two critical points.

So far the above conjecture has not been proved or denied. It is still open.

In our opinion, we should start to study period function for quadratic integrable systems whose orbits are algebraic curves of low degree. It should be easier to deal with than the general cases. Indeed, some results have been obtained in this direction. The authors of the paper [30] studied the quadratic centers whose almost all orbits are conics. They proved that such center must be an isochronous one or linear center after removing the linear factor. In the sequel, the phrase “almost all” means “all except at most a finite number.”

In this paper, we study the monotonicity of period function of periodic orbits for quadratic integrable systems with centers whose almost all orbits are cubics. Using the results from [27] and taking a complex coordinate  $z = x + iy$ , the list of quadratic centers at  $(0, 0)$  having almost all their orbits are cubics looks as follows:

(i) The Hamiltonian system  $Q_3^H$ :

$$\begin{aligned}\dot{z} &= -iz - z^2 + 2|z|^2 + (b + ic)\bar{z}^2, \\ H(x, y) &= \frac{1}{2}(x^2 + y^2) + \left(\frac{b}{3} - 1\right)x^3 + cx^2y - (1 + b)xy^2 - \frac{c}{3}y^3.\end{aligned}\quad (1.3)$$

(ii) The Hamiltonian triangle:

$$\begin{aligned}\dot{z} &= -iz + \bar{z}^2, \\ H(x, y) &= (1 - 2x)\left[\frac{1}{2}y^2 - \frac{1}{6}(x + 1)^2\right].\end{aligned}\quad (1.4)$$

(iii) The reversible system:

$$\begin{aligned}\dot{z} &= -iz + (2b + 1)z^2 + 2|z|^2 + b\bar{z}^2, \quad b \neq -1, \\ H(x, y) &= X^{-3}\left[\frac{1}{2}y^2 + \frac{1}{8(b+1)^2}\left(\frac{1-3b}{b+1}X^2 + 2\frac{b-1}{b+1}X + \frac{3-b}{3b+3}\right)\right],\end{aligned}\quad (1.5)$$

where  $X = 1 + 2(b + 1)x$  and  $M(x, y) = X^{-4}$ .

(iv) The generic Lotka–Volterra system:

$$\begin{aligned}\dot{z} &= -iz + (1 - ci)z^2 + ci\bar{z}^2, \quad c = \pm \frac{2}{\sqrt{3}}, \\ H(x, y) &= -\frac{1}{2}(1 + 2x)(1 + x \pm \sqrt{3}y)\left(1 + x \pm \frac{\sqrt{3}}{3}y\right)^{-3}, \\ M(x, y) &= \left(1 + x \pm \frac{\sqrt{3}}{3}y\right)^{-4}.\end{aligned}\quad (1.6)_{\pm}$$

System (1.3) and (1.4) are Hamiltonian quadratic systems. Coppel and Gavrilov [15] have shown that the period function of quadratic Hamiltonian systems is monotone. In this paper, we only consider system (1.5) and (1.6)<sub>±</sub>. To convenience, we give another normal form of system (1.5) and (1.6)<sub>±</sub> respectively. More precisely, we have

**Lemma 1.1.** *By changes*

$$x \rightarrow \alpha x + \beta y, \quad y \rightarrow \gamma x + \eta y, \quad t \rightarrow \sigma t, \quad (1.7)$$

system (1.5) can be reduced to

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = A + 2 - 2(A + 1)x + Ax^2 + \frac{3}{2}y^2, \end{cases} \quad (1.8)$$

where  $\alpha, \beta, \gamma, \eta, \sigma$  are real numbers. A first integral of system (1.8) is given by

$$H(x, y) = x^{-3} \left[ \frac{1}{2}y^2 + Ax^2 - (A + 1)x + \frac{A + 2}{3} \right] = h \quad (1.9)$$

with the integrating factor  $M(x, y) = x^{-4}$ . Moreover, we have

- (i) If  $A < -3$ , then system (1.8) has a center at  $P_1(1, 0)$ , a saddle at  $P_2((A + 2)/A, 0)$ , a stable node at  $(0, -\sqrt{-2(A + 2)/3})$  and an unstable node at  $(0, \sqrt{-2(A + 2)/3})$ . The oval  $\Gamma_h$  around  $P_1$  is defined for Hamiltonian value  $h \in \Sigma = (h_1, h_2)$ , where

$$h_1 = H(1, 0) = \frac{A - 1}{3}, \quad h_2 = H\left(\frac{A + 2}{A}, 0\right) = \frac{A^2(A + 3)}{3(A + 2)^2}.$$

- (ii) If  $-3 \leq A < -2$ , then system (1.8) has four critical points as case (i), the oval  $\Gamma_h$  around  $P_1$  is defined in  $\Sigma = (h_1, 0)$ .
- (iii) If  $A = -2$ , then system (1.8) has a center at  $P_1$  and a degenerate critical points at  $(0, 0)$ . The period orbits are defined in  $\Sigma = (-1, 0)$ .
- (iv) If  $-2 < A < 0$ , then system (1.8) has two centers at  $P_1$  and  $P_2$ , respectively. The period orbits around  $P_i$  are defined in  $\Sigma_i$ ,  $i = 1, 2$ , where  $\Sigma_1 = (h_1, 0)$ ,  $\Sigma_2 = (0, h_2)$ .
- (v) If  $A = 0$ , then system (1.8) has a unique critical point at  $P_1$  which is a center. The oval  $\Gamma_h$  is defined in  $\Sigma = (-1/3, 0)$ .
- (vi) If  $A > 0$ , then system (1.8) has a center at  $P_1$  and a saddle at  $P_2$ . The period orbit  $\Gamma_h$  is defined in  $h \in \Sigma = (h_1, h_2)$ .

**Lemma 1.2.** System (1.6) $_{\pm}$  may be brought into the following normal form:

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = -\frac{3}{8} + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{3}{2}y^2 \end{cases} \quad (1.10)$$

by using the linear transformation as (1.7). A first integral of system (1.10) is given by

$$H(x, y) = x^{-3} \left( \frac{1}{2}y^2 - \frac{9}{8}x^2 + \frac{3}{4}x - \frac{1}{8} \right) = h \quad (1.11)$$

with integrating factor  $M(x, y) = x^{-4}$ . System (1.10) has a center at  $(1, 0)$ , a stable node at  $(0, -1/2)$  an unstable node at  $(0, 1/2)$  and a saddle at  $(1/3, 0)$ , respectively. The period orbit  $\Gamma_h$  around the center  $(1, 0)$  is defined in  $\Sigma = (-1/2, 0)$ .

The main results of this paper are the following theorems.

**Theorem 1.3.** Let  $A \in (-\infty, +\infty)$  and

$$\begin{aligned} A_1 &= \frac{-7 - 2\sqrt{6}}{5} \approx -2.38, & A_2 &= \frac{-7 + 2\sqrt{6}}{5} \approx -0.42, \\ \tilde{A}_1 &= \frac{-3 - 2\sqrt{6}}{5} \approx -1.58. \end{aligned}$$

Here  $A_1, A_2$  are zeros of  $5A^2 + 14A + 5$  and  $\tilde{A}_1$  is a zero of  $5A^2 + 6A - 3$ .

- (i) If  $A \in (-3, A_1) \cup (A_2, 0)$ , then the period function of periodic orbits around the center  $P_1(1, 0)$  has a unique critical point. For any case of others, it is monotone.
- (ii) The period of periodic orbits around the center  $P_2((A+2)/A, 0)$  is monotone for  $A \in [\tilde{A}_1, 0)$  and has a unique critical point for  $A \in (-2, \tilde{A}_1)$ .

**Theorem 1.4.** *The period function for  $(1.6)_\pm$  is monotone.*

The above two theorems together with the results for quadratic Hamiltonian system from [15] yield

**Theorem 1.5.** *For the quadratic integrable system with centers whose almost all orbits lie on cubic planar curves, the corresponding period function has at most one critical point.*

It is remarkable that so many different methods have been applied to study the period function, see [4,9–12,18,22,31,36,40,45,48,49] and references therein. The proof in this paper is based on Picard–Fuchs equation for algebraic curves. It has been used by many authors in the study of the number of zeros of Abelian integrals, cf. [19–21,25–27,33–35,53] etc. We also note that the study of period function is important in the analysis of nonlinear boundary value problem [2,43] and in the study of the weakened Hilbert 16th problem [16].

## 2. Differential equations related to period function

In this section, we will express the period function as an Abelian integral and derive some differential equations related to period function. To do it, define

$$J_i(h) = \oint_{\Gamma_h} M(x, y)x^i y dx = \oint_{\Gamma_h} x^{i-4} y dx, \quad i = \dots, -1, 0, 1, \dots, \quad (2.1)$$

where  $\Gamma_h$  is defined in Lemma 1.1 or 1.2. In the following the prim' denotes the derivative with respect to  $h$ . It follows from (1.9) (respectively (1.11)) that

$$\frac{\partial y}{\partial h} = \frac{x^3}{y}, \quad (2.2)$$

which implies that

$$J'_i(h) = \oint_{\Gamma_h} \frac{x^{i-1}}{y} dx. \quad (2.3)$$

It follows from (1.2), (1.8) (respectively (1.10)) and (2.3) that

$$T(h) = J'_0(h). \quad (2.4)$$

Therefore, the period functions  $T(h)$  for system (1.8) and (1.10) is the derivative of Abelian integral  $J_0(h)$ .

In a standard way (see [26,27], for example), we get in [52] that

**Lemma 2.1.** *Let  $B = -A - 1$ ,  $C = (A + 2)/3$ . Suppose  $A \neq -2$ , then the vector  $\mathbf{V} = \text{col}(J_{-1}(h), J_0(h), J_1(h))$ , associated with system (1.8), satisfies the following Picard–Fuchs equation:*

$$(\mathbf{D}h + \mathbf{S})\mathbf{V}' = \mathbf{U}\mathbf{V}, \quad (2.5)$$

where

$$\begin{aligned} \mathbf{D} &= \begin{pmatrix} -6C^2 & 2BC & 2(2AC - B^2) \\ 0 & -6C & 2B \\ 0 & 0 & -3 \end{pmatrix}, \\ \mathbf{S} &= \begin{pmatrix} 2B(B^2 - 3AC) & 2A(B^2 - 2AC) & 0 \\ 2(2AC - B^2) & -2AB & 0 \\ B & 2A & 0 \end{pmatrix}, \\ \mathbf{U} &= \begin{pmatrix} -8C^2 & BC & 2AC - B^2 \\ 0 & -6C & B \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

Differentiating both sides of (2.5), we get

$$(\mathbf{D}h + \mathbf{S})\mathbf{V}'' = (\mathbf{U} - \mathbf{D})\mathbf{V}', \quad (2.6)$$

where

$$\mathbf{U} - \mathbf{D} = \begin{pmatrix} -2C^2 & -BC & B^2 - 2AC \\ 0 & 0 & -B \\ 0 & 0 & 1 \end{pmatrix},$$

which contains zero sub-matrix of order two in the lower-left corner. Therefore, taking

$$Z = -2C^2 J_{-1} - BC J_0, \quad (2.7)$$

we get the following corollary from (2.6) and (2.7).

**Corollary 2.2.** *The following relation holds for system (1.8), provided  $A \neq -2$ :*

$$\Delta(h) \begin{pmatrix} Z'' \\ J_1'' \end{pmatrix} = \begin{pmatrix} a_{00}(h) & a_{01}(h) \\ a_{10}(h) & a_{11}(h) \end{pmatrix} \begin{pmatrix} Z' \\ J_1' \end{pmatrix}, \quad (2.8)$$

and

$$\Delta(h) J_0'' = h \left\{ 4Z' + \frac{2}{9}(A+2)[A(A+3)(A-1) - 3(A+1)(A+2)h]J_1' \right\}, \quad (2.9)$$

where

$$\begin{aligned}
\Delta(h) &= -4(A+2)^3 h(h-h_1)(h-h_2), \\
a_{00}(h) &= \frac{4}{9}(A+2)h\{(A-1)(A+1)(A+3) - 3(A+2)^2 h\}, \\
a_{01}(h) &= -\frac{2}{27}(A+2)^2 h\{A(A-1)(A+1)(A+3) - (A+2)(3A^2 + 6A - 1)h\}, \\
a_{10}(h) &= \frac{2}{3}\{A(A-1)(A+3) - 3(A+1)(A+2)h\}, \\
a_{11}(h) &= -a_{00}(h).
\end{aligned}$$

Noting that  $x = 0$  is an invariant line of system (1.8), we conclude that the period orbit  $\Gamma_h$  does not intersect  $y$ -axis, which means that  $x \neq 0$  for  $x \in \Gamma_h$ . Suppose that  $(x(t), y(t))$  is a solution of system (1.8) with period  $T(h)$ , then it follows from (2.3) and the first equation of (1.8) that

$$J_1'(h) = \int_0^{T(h)} \frac{1}{y} dx = \int_0^{T(h)} \frac{x(t)y(t)}{y(t)} dt = \int_0^{T(h)} x(t) dt \neq 0.$$

Hence, we can define

$$\omega(h) = \frac{Z'(h)}{J_1'(h)}. \quad (2.10)$$

The main results of this section are the following three propositions.

**Proposition 2.3.** Suppose  $A \neq -2$  in (1.8), then

- (i) The number of critical point of period function  $T(h)$ , related to system (1.8), is equal to the number of zeros of  $\phi(h)$ , defined by

$$\phi(h) = \frac{2}{9}(A+2)[A(A+3)(A-1) - 3(A+1)(A+2)h] + 4\omega(h). \quad (2.11)$$

- (ii)  $\phi(h)$  satisfies the following Riccati equation:

$$\Delta(h)\phi' = R_0(h)\phi^2 + R_1(h)\phi + R_2(h), \quad (2.12)$$

where

$$\begin{aligned}
R_0(h) &= -\frac{1}{4}a_{10}(h), \\
R_1(h) &= \frac{2}{3}(A-1)(A+2)^3(A+3)(h-h_1)(h-h_2), \\
R_2(h) &= -\frac{2}{27}(A+2)^4(h-h_1)(h-h_2)[A(A-1)^2(A+3)^2 \\
&\quad - 3(A+1)(A+2)(A^2 + 2A + 5)h].
\end{aligned}$$

**Proof.** The result (i) follows from (2.4) and (2.9). Since

$$\omega'(h) = \frac{Z''J_1' - Z'J_1''}{J_1'^2},$$

we get from (2.8) that  $\omega(h)$  satisfies

$$\Delta(h)\omega' = -a_{10}(h)\omega^2 + 2a_{00}(h)\omega + a_{01}(h). \quad (2.13)$$

Substituting (2.11) into (2.13), we obtain (2.12).  $\square$

**Proposition 2.4.** *If  $A = -2$ , then the ratio  $u(h) = T'(h)/T(h)$  satisfies the following equation:*

$$16h(h+1)u' = -16h(h+1)u^2 - 16hu + 1, \quad (2.14)$$

where  $T(h)$  is the period function associated with system (1.8).

**Proof.** In the paper [52], we have shown that  $J_{-1}(h)$ ,  $J_0(h)$ , associated with system (1.8), satisfy the following Picard–Fuchs equation, provided  $A = -2$ :

$$4h(h+1) \begin{pmatrix} J_{-1}' \\ J_0' \end{pmatrix} = \begin{pmatrix} 7h & -3h \\ -7 & 5h+8 \end{pmatrix} \begin{pmatrix} J_{-1} \\ J_0 \end{pmatrix}. \quad (2.15)$$

It follows from the second equation of system (2.15) that

$$J_{-1} = \frac{1}{7}[(5h+8)J_0 - 4h(h+1)J_0']. \quad (2.16)$$

Substituting (2.16) into the first equation of system (2.15), we get

$$16h(h+1)J_0'' = 16(h+1)J_0' - 15J_0. \quad (2.17)$$

Differentiating both side of (2.17), we have

$$16h(h+1)J_0''' = -16hJ_0'' + J_0'.$$

Substituting (2.4) into the above equation, we have

$$16h(h+1)T'' = -16hT' + T, \quad (2.18)$$

which implies (2.14).  $\square$

**Proposition 2.5.**  $J_0(h)$ , related system (1.10), satisfies the following equation:

$$\frac{9}{2}h(1+2h)J_0'' = -2J_0. \quad (2.19)$$

**Proof.** For system (1.10), we have proved in [52] that  $J_{-1}(h)$ ,  $J_0(h)$  satisfy

$$3h(1+2h) \begin{pmatrix} J_{-1}' \\ J_0' \end{pmatrix} = \begin{pmatrix} 2(4h+3) & -8h-9 \\ 2 & 4h-3 \end{pmatrix} \begin{pmatrix} J_{-1} \\ J_0 \end{pmatrix}. \quad (2.20)$$

By the second equation of system (2.20),  $J_{-1}(h)$  can be expressed as a linear combination of  $J_0(h)$ ,  $J_0'(h)$ . Substituting this expression into the first equation of system (2.20), we get (2.19).  $\square$

We end this section by the following lemmas.



**Lemma 2.6.** Suppose  $A \neq -2$  in (1.8), then  $\phi(h_1) = \phi(h_2) = 0$ ,  $\phi'(h_1) = -(A+2)(5A^2 + 14A + 5)/9$ ,  $\phi(0) = 2A(A+2)(A+3)(A-1)/9$ . Moreover, if  $\Gamma_{h_2}$  corresponds to the center of system (1.8), i.e., in the case  $-2 < A < 0$ , then  $\phi'(h_2) = -(A+2)^2(5A^2 + 6A - 3)/(9A)$ ; if  $\Gamma_{h_2}$  corresponds to the saddle-loop of system (1.8), then

$$\phi(h) = -\frac{2}{3}(A+1)(A+2)^2(h-h_2) + \frac{16A}{3\ln(h_2-h)} + \cdots \quad (2.21)$$

as  $h \rightarrow h_2$ . Here we define  $\phi(h_1)$  by  $\phi(h_1) = \lim_{h \rightarrow h_1} \phi(h)$ .

**Proof.** Since the Hamiltonian value  $h = h_1$  corresponds to the center  $P_1(1, 0)$  of system (1.8), we conclude that  $J_1(h)$  and  $Z(h)$  are analytic at  $h = h_1$  [29,39]. Taking  $h = h_1$  into the first equation of (2.8), we get  $\omega(h_1) = (A-1)(A+2)/9$ . Substituting  $\omega(h_1)$  into (2.11), we have  $\phi(h_1) = 0$ . Since  $J_1(h)$ ,  $Z(h)$  are analytic and  $\omega(h_1)$  is finite at  $h = h_1$ , we conclude that  $\phi(h)$  is analytic at  $h = h_1$ , too. Differentiating both sides of (2.12) and then substituting  $h = h_1$ ,  $\phi(h_1) = 0$  into it, we obtain  $\phi'(h_1)$ .

It is easy to see that  $h = h_2$  is a simple singularity [23] of system (2.8). Using analytic theory of ordinary differential equations (cf. Appendix of Chapter IV in [23]), we get a fundamental matrix solution of system (2.8) near  $h = h_2$ :

$$\begin{pmatrix} A(A+3)/9 & A(A+15)/9 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+o(1) & (1+o(1))\ln(h_2-h) \\ 0 & 1+o(1) \end{pmatrix},$$

where  $o(1)$  are analytic function of  $h$  and  $\lim_{h \rightarrow h_2} o(1) = 0$ . This implies

$$\begin{aligned} Z' &= \frac{A(A+3)}{9}C_2 \ln(h_2-h) + \frac{A(A+15)}{9}C_2 + \frac{A(A+3)}{9}C_1 + \cdots, \\ J_1' &= C_2 \ln(h_2-h) + C_1 + C_2 + \cdots, \end{aligned} \quad (2.22)$$

where  $C_1$  and  $C_2$  are real constants. This shows  $\omega(h_2) = A(A+3)/9$ . Using (2.11) again, we get  $\phi(h_2) = 0$ . To get (2.21), consider

$$\dot{x} = H_y(x, y), \quad \dot{y} = -H_x(x, y) + \epsilon x^{-3}y.$$

In the right half plane  $\{(x, y) \mid x > 0\}$ , the above system is a one parameter analytic system. Suppose that  $\Gamma_h$  has a negative (clockwise) orientation, then  $J_1(h)$  can be expressed as the following form (cf. [29] or [53, Appendix]):

$$J_1(h) = \oint_{\Gamma_h} x^{-3}y \, dx = J_1(h_2) - \sqrt{\frac{A+2}{2A}} \ln(h_2-h) + \cdots,$$

as  $h \rightarrow h_2$ , which implies  $C_2 \neq 0$  in the expansion of  $J_1'(h)$ . Hence,

$$\omega(h) = \frac{Z'(h)}{J_1'(h)} = \frac{A(A+3)}{9} + \frac{4A}{3\ln(h_2-h)} + \cdots. \quad (2.23)$$

Using (2.11) and (2.23), the expansion (2.21) follows.

If  $\Gamma_{h_2}$  corresponds to a center, then by the same arguments as in the case  $h = h_1$ , we get  $\phi'(h_2)$ .

Near  $h = 0$ , we have a fundamental matrix solution in the form

$$\begin{pmatrix} 0 & -2A(A+2)/3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1+o(1) & (1+o(1))\ln|h| \\ o(1) & 1+o(1)+o(1)\cdot\ln|h| \end{pmatrix},$$

which means

$$\begin{aligned} Z' &= -\frac{2A(A+2)}{3}C_2' + o(1), \\ J_1' &= C_2' \ln|h| + C_1' + C_2' + o(1), \end{aligned}$$

where  $C_1', C_2'$  are real constants and  $\lim_{h \rightarrow 0} o(1) = 0$ . Therefore,  $\omega(0) = 0$ . Finally, we obtain  $\phi(0)$  from (2.11).  $\square$

**Lemma 2.7.** *If  $A = -2$ , then  $u(-1) = -1/16$ , where  $u(h)$  is defined in Proposition 2.4.*

**Proof.** Since  $h = -1$  corresponds to the center  $(1, 0)$  of system (1.8), we know that  $J_0(h)$  is analytic at  $h = -1$  [29,39]. Taking  $h = -1$  into (2.18), we have  $u(-1) = -1/16$ .  $\square$

### 3. Proof of Theorem 1.4 and Theorem 1.3 for $A \notin (0, 1)$

In this section, we will prove the main results for  $A \notin (0, 1)$ .

**Proof of Theorem 1.4.** To study the period of system (1.6) $_{\pm}$ , we consider the equivalent system (1.10). Since  $x = 0$  is an invariant line, the period orbit  $\Gamma_h$  around the center  $(1, 0)$  lies on the right half phase plane, which implies  $x > 0$  for  $x \in \Gamma_h \cup \text{int } \Gamma_h$ . By Green formula,

$$J_0(h) = \oint_{\Gamma_h} x^{-4} y dx = \int \int_{\text{int } \Gamma_h} x^{-4} dx dy \neq 0.$$

It follows from (2.19) that  $J_0''(h) \neq 0$  for  $h \in \Sigma = (-1/2, 0)$ . Using (2.4) again, we have  $T'(h) \neq 0$ ,  $h \in \Sigma$ . The proof is finished.  $\square$

**Proof of Theorem 1.3 for  $A = -2$ .** It is obvious that  $u(h) = T'(h)/T(h) = 0$  if and only if  $T'(h) = 0$ . By Lemma 2.7,  $u(-1) = -1/16 < 0$ . Starting from  $h = -1$ , suppose that  $h = h^*$  is the first zero of  $u(h)$ , i.e.,  $u(h^*) = 0$ ,  $u(h) \neq 0$  for  $h \in (-1, h^*)$ ,  $-1 < h^* < 0$ . Since  $u(-1) < 0$ , we have  $u'(h^*) \geq 0$ . However, it follows from (2.14) that  $u'(h^*) = 1/(16h^*(h^* + 1)) < 0$ , which yields contradiction. Hence,  $u(h) \neq 0$  in  $\Sigma = (-1, 0)$ , i.e.,  $T'(h) \neq 0$ .  $\square$

By Proposition 2.3, we will estimate the number of zeros of  $\phi(h)$  instead of  $T'(h)$ .

**Lemma 3.1.** *Suppose  $A \in (-\infty, -3] \cup [A_1, -2) \cup (-2, A_2] \cup \{0\} \cup [1, +\infty)$ , then the period function  $T(h)$  of periodic orbits around the center  $P_1(1, 0)$  is monotone.*

**Proof.** Let  $h^*$  be the zero of  $\phi(h)$ ,  $h^* \in \Sigma$ . It follows from (2.12) that

$$\phi'(h^*) = \frac{R_2(h^*)}{\Delta(h^*)} = \frac{(A+2)g(h^*)}{54h}, \quad (3.1)$$

where

$$g(h) = A(A-1)^2(A+3)^2 - 3(A+1)(A+2)(A^2+2A+5)h, \quad (3.2)$$

which implies

$$\begin{aligned} g(h_1) &= -2(A-1)(5A^2+14A+5), & g(0) &= A(A-1)^2(A+3)^2, \\ g(h_2) &= -\frac{2A(A+3)(5A^2+6A-3)}{A+2}. \end{aligned} \quad (3.3)$$

By Lemma 2.6,  $\phi(h_1) = 0$ . Suppose that  $h_1$  and  $h^*$  are two consecutive zeros of  $\phi(h)$ , i.e.,  $\phi(h^*) = 0$ ,  $\phi(h) \neq 0$  for  $h_1 < h < h^*$ , then we have  $\phi'(h_1)\phi(h^*) \leq 0$ .

If  $A \in (-\infty, -3]$ , then  $\Gamma_h$  is defined in  $\Sigma = (h_1, h_2)$  and  $\phi'(h_1) > 0$ ,  $g(h_1) > 0$ ,  $g(h_2) \geq 0$ ,  $h_2 < 0$ . Since  $g(h)$  is a linear function of  $h$ , we get  $g(h) > 0$  for  $h \in \Sigma$ , which implies  $\phi'(h^*) > 0$  in (3.1). Hence,  $\phi'(h_1)\phi'(h^*) > 0$ , which yields contradiction. By Proposition 2.3(i), the period function is monotone.

If  $A \in (A_1, -2)$ , then  $\Gamma_h$  is defined in  $\Sigma = (h_1, 0)$  and  $\phi'(h_1) < 0$ ,  $g(0) < 0$ ,  $g(h_1) < 0$ . Using the same arguments as above, we have  $\phi'(h_1)\phi'(h^*) > 0$ , which yields contradiction, too.

If  $A = A_1$ , then  $\phi(A_1, h_1) = \phi'(A_1, h_1) = 0$ , where  $\phi(A_1, h_1) = \phi|_{A=A_1}(h_1)$ . Differentiating both sides of (2.12) twice, we get

$$\phi''(A_1, h_1) = \frac{-4(-3+2\sqrt{6})^2(22+17\sqrt{6})}{1875(6+\sqrt{6})} < 0,$$

which shows  $\phi'(A_1, h) = 2\phi''(A_1, h_1)(h-h_1) + \dots < 0$  as  $h \rightarrow h_1$ ,  $h \in (h_1, 0)$ . Therefore,  $\phi'(A_1, h^*) \geq 0$  must hold if  $h_1, h^*$  are two consecutive zeros of  $\phi(h)$ . Since  $g(A_1, h_1) = 0$ ,  $g(A_1, 0) < 0$  implies  $g(A_1, h) < 0$  for  $h \in (h_1, 0)$ , we have  $\phi'(A_1, h^*) < 0$ . This yields contradiction.

If  $A = A_2$ , then  $\phi(A_2, h_1) = \phi'(A_2, h_1) = 0$  and

$$\phi''(A_1, h_1) = \frac{-4(3+2\sqrt{6})^2(-22+17\sqrt{6})}{1875(-6+\sqrt{6})} > 0.$$

Using the same arguments as the case  $A = A_1$ , we get  $T'(h) \neq 0$  in  $\Sigma = (h_1, 0)$ .

If  $A \in (-2, A_2)$ , then it follows from (3.1)–(3.3) that  $\phi'(h^*) > 0$  for  $h^* \in \Sigma_1 = (h_1, 0)$ . Since  $\phi'(h_1) > 0$  in this case, this yields contradiction. The period function of periodic orbits around the center  $P_1(1, 0)$  is monotone.

If  $A = 0$ , then  $\Sigma = (-1/3, 0)$ ,  $\phi'(h_1) = -10/9$  and  $\phi'(h^*) = -10/9$ , which yields contradiction. Hence,  $\phi(h) \neq 0$ . By Proposition 2.3(i),  $T'(h) \neq 0$ .

If  $A \in [1, +\infty)$ , then  $\Sigma = (h_1, h_2)$  and  $\phi'(h_1) < 0$ ,  $g(h_1) \leq 0$ ,  $g(h_2) < 0$ ,  $h_1 \geq 0$ . By the same arguments, we get the result.  $\square$

**Lemma 3.2.** If  $A \in (-3, A_1) \cup (A_2, 0)$ , then  $\phi(h)$  has a unique zero in  $h \in (h_1, 0)$ .

**Proof.** For any two consecutive zeros  $h_1^*, h_2^*$  of  $\phi(h)$ , we have  $\phi'(h_1^*)\phi(h_2^*) \leq 0$ . Since  $g(h)$  is a linear function of  $h$  with one zero in  $(h_1, 0)$  for  $A \in (-3, A_1) \cup (A_2, 0)$ , it follows from (3.1) that  $\phi(h)$  has at most two zeros in  $(h_1, 0)$ . If  $A \in (-3, A_1)$ , then  $\phi'(h_1) > 0$ , which yields that  $\phi(h) = \phi'(h_1)(h - h_1) + \dots > 0$  as  $h \rightarrow h_1, h \in (h_1, 0)$ . Since  $\phi(0) < 0$ , we conclude that the number of zeros of  $\phi(h)$  must be odd. Note that  $\phi(h)$  has at most two zeros, it follows that  $\phi(h)$  has a unique zero in  $(h_1, 0)$ .

If  $A \in (A_2, 0)$ , then  $\phi(0) > 0$  and  $\phi(h) = \phi'(h_1)(h - h_1) + \dots < 0$  as  $h \rightarrow h_1, h \in (h_1, 0)$ . Using the same arguments as above, we know that  $\phi(h)$  has a unique zero in  $(h_1, 0)$ .  $\square$

**Lemma 3.3.** Suppose that  $A \in (-2, \tilde{A}_1)$ , then the period function  $T(h)$  of period orbits around  $P_2((A+2)/A, 0)$  has one critical point.

**Proof.** In this case,  $g(h)$  has one zero in  $(0, h_2)$ . By the same arguments as Lemma 3.2, we know that  $\phi(h)$  has at most two zeros. Since  $\phi'(h_2) > 0$  implies  $\phi(h) = \phi'(h_2)(h - h_2) + \dots < 0$  as  $h \rightarrow h_2, h \in \Sigma_2 = (0, h_2)$ , it follows from  $\phi(0) > 0$  that  $\phi(h)$  has a unique zero in  $\Sigma_2$ , i.e.,  $T(h)$  has one critical point.  $\square$

**Lemma 3.4.** Suppose  $A \in [\tilde{A}_1, 0)$ , then the period function  $T(h)$  of periodic orbits around  $P_2((A+2)/A, 0)$  is monotone.

**Proof.** In this case,  $T(h)$  is defined in  $\Sigma_2 = (0, h_2)$  and  $g(h) < 0$  for  $h \in \Sigma_2$ . If  $A \in (\tilde{A}_1, 0)$ , then  $\phi'(h_2) < 0$ . If  $A = \tilde{A}_1$ , then  $\phi(\tilde{A}_1, h_2) = \phi'(\tilde{A}_1, h_2) = 0$  and

$$\phi''(\tilde{A}_1, h_1) = \frac{-4(-7 + 2\sqrt{6})^4(-22 + 17\sqrt{6})}{1875(-6 + \sqrt{6})(3 + 2\sqrt{6})^2} > 0,$$

which implies  $\phi'(\tilde{A}_1, h) = 2\phi''(\tilde{A}_1, h_2)(h - h_2) + \dots < 0$  as  $h \rightarrow h_2, h \in (0, h_2)$ . Using the same arguments as Lemma 3.1, we get the result.  $\square$

#### 4. Proof of Theorem 1.3 for $A \in (0, 1)$

If  $A \in (0, 1)$ , then  $\Sigma = (h_1, 0) \cup [0, h_2)$ . In this case, we can obtain that  $\phi(h)$  has at most two zeros by the same arguments as in the last section. It is not enough for our proof. To show the monotonicity of  $T(h)$ , we must use other method. This will be done by argument principle. In this section, we suppose that  $A \in (0, 1)$  unless the opposite is claimed.

Let  $\tilde{J}_1(h), \tilde{Z}(h)$  be analytic continuations of  $J_1(h), Z(h)$  from  $\Sigma = (h_1, h_2)$  to the whole complex domain  $\mathbb{C}$ . Since  $h = h_1$  and  $h = 0$  correspond to the center and periodic orbit of system (1.8), respectively,  $J_1(h)$  and  $Z(h)$  are analytic at  $h = h_1$  and  $h = 0$  [29,39]. System (2.8) is a linear system with simple singular points, which means that its solutions, including  $\text{col}(\tilde{Z}', \tilde{J}_1')$ , are (multiply valued) analytic functions on  $\mathbb{C} \setminus \{h = h_2\}$ . To get the single-valued function on  $\mathbb{C}$ , define

$$\mathcal{D} = \mathbb{C} \setminus \{h \in [h_2, +\infty), h \in \mathbb{R}\}.$$

By above discussions, we conclude that  $\tilde{J}'_1, \tilde{Z}'$  are single-valued analytic function on  $\mathcal{D}$ .

We will estimate the number of zeros of  $\phi(h)$  by argument principle. To do it, we have to show that  $\tilde{\omega}(h)$  is analytic on  $\mathcal{D}$ , where

$$\tilde{\omega}(h) = \frac{\tilde{Z}'(h)}{\tilde{J}'_1(h)}. \quad (4.1)$$

In the first step, we get

**Lemma 4.1.** *Let  $S_+$  (respectively  $S_-$ ) be the upper (respectively lower) side of the open cut  $\{h \in (h_2, +\infty), h \in \mathbb{R}\}$ . Then for  $h \in S_{\pm}$ , we have  $\tilde{J}'_1(h) \neq 0$ ,  $\text{Im } \tilde{\omega}(h) \neq 0$ .*

**Proof.** Denote by  $\text{Im } \tilde{J}'_1, \text{Re } \tilde{J}'_1$  the imaginary part and the real part of  $\tilde{J}'_1$ , respectively. Since system (2.8) is a real analytic linear differential equations for  $h \in S_{\pm}$ , the vector  $\text{col}(\text{Im } \tilde{Z}', \text{Im } \tilde{J}'_1)$  and  $\text{col}(\text{Re } \tilde{Z}', \text{Re } \tilde{J}'_1)$  are analytic solutions of (2.8), respectively. By Liouville's formula, it follows from  $a_{11}(h) = -a_{00}(h)$  in (2.8) that the Wronskian determination

$$W(h) = \begin{vmatrix} \text{Re } \tilde{J}'_1 & \text{Im } \tilde{J}'_1 \\ \text{Re } \tilde{Z}' & \text{Im } \tilde{Z}' \end{vmatrix} = \text{constant}, \quad h \in S_{\pm}. \quad (4.2)$$

On the other hand,

$$\text{Im } \tilde{\omega}(h) = \frac{W(h)}{|\tilde{J}'_1(h)|^2}. \quad (4.3)$$

If there exists  $h = \beta$  such that  $\tilde{J}'_1(\beta) = 0$ , i.e.,  $\text{Im } \tilde{J}'_1(\beta) = \text{Re } \tilde{J}'_1(\beta) = 0$ , then  $\text{Im } \tilde{\omega}(h) \equiv 0$  by using (4.2) and (4.3). It follows from (2.23) that  $\text{Im } \tilde{\omega}(h) \neq 0$  near  $h = h_2, h > h_2$ , which yields contradiction.

Using (4.2), (4.3) and (2.23) again, we get  $\text{Im } \tilde{\omega}(h) \neq 0$  for  $h \in S_{\pm}$ .  $\square$

To prove  $\tilde{J}'_1(h) \neq 0$  on  $\mathbb{C}$ , we consider

$$v(h) = \frac{\text{Im } \tilde{Z}'}{\text{Im } \tilde{J}'_1}, \quad h \in S_{\pm}. \quad (4.4)$$

By the same arguments as in the proof of Lemma 4.1, we know that  $(\text{Im } \tilde{Z}')^2 + (\text{Im } \tilde{J}'_1)^2 \neq 0$ . Since  $\text{col}(\text{Im } \tilde{Z}', \text{Im } \tilde{J}'_1)$  is a solution of system (2.8), the ratio  $v(h)$  satisfies the Riccati equation (2.13). That is to say,  $v(h)$  is a trajectory of system

$$\begin{cases} \dot{h} = \Delta(h), \\ \dot{\omega} = -a_{10}(h)\omega^2 + 2a_{00}\omega + a_{01}(h). \end{cases} \quad (4.5)$$

System (4.5) has three saddle-node at  $(h_1, (A-1)(A+2)/9)$ ,  $(h_2, A(A+3)/9)$  and  $(0, 0)$ , respectively. The vertical zero isoclines  $h = h_1, h = h_2$  and  $h = 0$  are invariant lines of system (4.5). By direct computation,

$$\begin{aligned} & a_{00}^2(h) + a_{01}(h)a_{10}(h) \\ &= \frac{16}{9}(A+2)^6 h(h-h_1)(h-h_2) \left[ h - \frac{(A-1)(A+1)(A+3)}{4(A+2)^2} \right] > 0 \end{aligned}$$

for  $h \in (h_2, +\infty)$ , which means there exists two zero isoclines  $\omega^+(h)$  and  $\omega^-(h)$ ,  $h \in (h_2, +\infty)$ , defined by

$$\Psi(h, \omega) = -a_{10}(h)\omega^2 + 2a_{00}\omega + a_{01}(h) = 0,$$

where

$$\omega^\pm(h) = \frac{a_{00} \mp \sqrt{a_{00}^2(h) + a_{01}(h)a_{10}(h)}}{a_{10}(h)},$$

on which the vector field is horizontal. Note  $a_{10}(h) < 0$  for  $h \in (h_2, +\infty)$ , we have  $\omega^+(h) > \omega^-(h)$ ,  $h \in (h_2, +\infty)$ .

**Lemma 4.2.**

- (i)  $\omega^\pm(h_2) = A(A+3)/9$ ,  $d\omega^+(h_2)/dh = +\infty$ ,  $d\omega^-(h_2)/dh = -\infty$ .
- (ii)  $\omega^\pm(h)$  has the following expansions as  $h \rightarrow +\infty$ :

$$\omega^+(h) = \frac{4(A+2)^2}{3(A+1)}h + \dots, \quad \omega^-(h) = \frac{1}{36}(3A^2 + 6A - 1) + \dots.$$

- (iii)  $d\omega^+(h)/dh > 0$ ,  $d\omega^-(h)/dh < 0$ ,  $h \in (h_2, +\infty)$ .

**Proof.** By direct computation, we get (i) and (ii).

It follows from (i) and (ii)  $\omega^+(h)$  is increasing near  $h = h_2$  and  $h = +\infty$ . Suppose that  $\omega^+(h)$  has a maximum in  $(h_2, +\infty)$ , then it must be followed by a minimum. Hence, there must exist  $\bar{\omega} \in (\omega^+(h_2), +\infty)$  such that the straight line  $\omega = \bar{\omega}$  intersects  $\omega^+(h)$  in at least three points. Since  $\Psi(h, \bar{\omega})$  is a polynomial of  $h$  with degree two, it intersects  $\omega = \bar{\omega}$  at most two points, which yields contradiction. Therefore,  $d\omega^+(h)/dh > 0$ ,  $h \in (h_2, +\infty)$ .

We have known in (i) that  $\omega^-(h)$  is decreasing near  $h = h_2$ . If we can show that  $\omega^-(h)$  is decreasing near  $h = +\infty$  and  $\omega^-(h_2) > \omega_\infty = \lim_{h \rightarrow +\infty} \omega^-(h) = (3A^2 + 6A - 1)/36$ , then the monotonicity of  $\omega^-(h)$  follows by the same arguments as above. In what follows we are going to prove  $d\omega^-(h)/dh < 0$  as  $h \rightarrow +\infty$  and  $\omega^-(h_2) > \omega_\infty$ .

Substituting  $\omega = \omega_\infty$  into  $\Psi(h, \omega)$ , we get

$$\begin{aligned} \Psi(h, \omega_\infty) &= \frac{1}{1944} [A(1-A)(A+3)(3A^2 + 6A - 1)^2 \\ &\quad + 3(A+1)(A+2)(49 - 44A + 14A^2 + 36A^3 + 9A^4)h]. \end{aligned}$$

By direct computation,

$$\begin{aligned} \Psi(h_1, \omega_\infty) &= \frac{(-1+A)(-7-2A+A^2)^2}{972} < 0, \\ \Psi(h_2, \omega_\infty) &= \frac{A(3+A)(1+6A+A^2)^2}{972(2+A)} > 0, \end{aligned}$$

which implies that  $\Psi(h, \omega_\infty)$  has one zero in  $(h_1, h_2)$ . Since  $\Psi(h, \omega_\infty)$  is a linear function of  $h$ , we conclude that  $\Psi(h, \omega_\infty) \neq 0$  in  $(h_2, +\infty)$ . This means that  $\omega^-(h)$ ,  $h \in (h_2, +\infty)$ , does not intersect the straight line  $\omega = \omega_\infty$ . On the other hand,  $\omega^-(h_2) - \omega_\infty = (A^2 +$

$6A + 1)/36 > 0$  shows that  $\omega^-(h_2) > \omega_\infty = \lim_{h \rightarrow +\infty} \omega^-(h)$ . If  $\omega^-(h)$  is increasing as  $h \rightarrow +\infty$ , then  $\omega^-(h)$  must intersect the straight line  $\omega = \omega_\infty$ , which yields contradiction. Hence,  $d\omega^-(h)/dh < 0$  as  $h \rightarrow +\infty$ . The proof is finished.  $\square$

**Lemma 4.3.**  $v'(h) > 0$ ,  $\lim_{h \rightarrow +\infty} v(h) = +\infty$ ,  $A(A + 3)/9 < \omega^-(h) < v(h) < \omega^+(h) < +\infty$ ,  $h \in S_\pm$ .

**Proof.** We only prove this lemma for  $h \in S_+$ . For the case  $h \in S_-$ , the same arguments can be used.

Consider system (4.5) and its trajectory  $v(h)$ ,  $h \in (h_2, +\infty)$ , defined in (4.4). In the phase plane of system (4.5), the region  $\{(h, \omega) \mid h \geq h_2\}$  is divided into three parts by  $\omega^\pm(h)$  and  $h = h_2$ . By (2.22),  $v(h_2) = A(A + 3)/9 = \omega^\pm(h_2)$  and  $-\infty = d\omega^-(h)/dh < v'(h_2) < d\omega^+(h)/dh = +\infty$ , which yields  $v(h)$  is a trajectory entering the critical  $(h_2, A(A + 3)/9)$  and  $\omega^-(h) < v(h) < \omega^+(h)$  as  $h \rightarrow h_2$ . Noting that  $\omega^+(h)$  and  $\omega^-(h)$  are monotonically increasing and decreasing, respectively, the trajectory  $v(h)$  must stay in  $\{(h, \omega) \mid \omega^-(h) < \omega < \omega^+(h)\}$ , see Fig. 1, which yields  $v'(h) > 0$  for  $h \in (h_2, +\infty)$  and  $\omega^-(h) < v(h) < \omega^+(h)$ . Since there is not any other critical point in  $\{(h, \omega) \mid h > h_2\}$ , we have  $\lim_{h \rightarrow +\infty} v(h) = +\infty$ , i.e., the trajectory  $v(h)$  starts from the critical point which is the intersection point of the equator ( $h = \infty$ ) and the zero isocline  $\omega^+(h)$ . The proof is finished.  $\square$

**Lemma 4.4.**  $\tilde{J}'_1(h) \neq 0$ ,  $h \in \mathcal{D}$ .

**Proof.** Suppose that there exists  $\alpha$  such that  $\text{Im } \tilde{J}'_1(\alpha) = 0$ , then it follows from Lemma 4.3 that  $\text{Im } \tilde{Z}'(\alpha) = 0$  (otherwise  $v(h) \rightarrow \infty$  as  $h \rightarrow \alpha$ ), which shows  $W(h) = W(\alpha) \equiv 0$  from (4.2). This implies  $\text{Im } \tilde{\omega}(h) \equiv 0$ ,  $h \in S_\pm$ , by (4.3), which contradicts Lemma 4.1. Hence,  $\text{Im } \tilde{J}'_1(h) \neq 0$  for  $h \in S_\pm$ .

In what follows we are going to prove  $\tilde{J}'_1(h) \neq 0$ ,  $h \in \mathcal{D}$ , by argument principle.

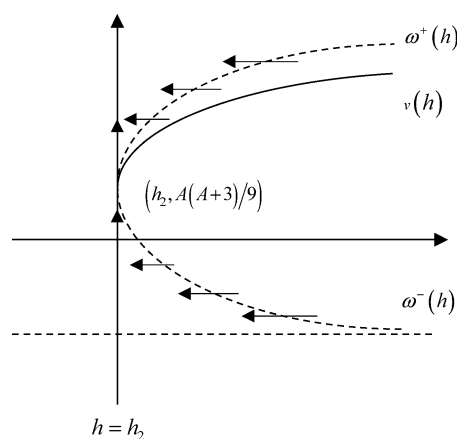


Fig. 1.

Let  $d_\infty$  be big enough and  $d_2$  be small enough constants. Denote by  $\mathcal{D}'$  the set obtained from  $\mathcal{D} \cap \{|h| < d_\infty\}$  by removing a circle of radius  $d_2$  around  $h = h_2$ . Consider the increase of argument of  $\tilde{J}'_1(h)$  along the boundary  $\partial\mathcal{D}'$  of  $\mathcal{D}'$  which is the positive orientation.  $\tilde{J}'_1(h)$  is a single valued analytic function in  $\mathcal{D} \supset \mathcal{D}'$ . Using analytic theory of ordinary differential equation [23], we have

$$\begin{aligned}\tilde{J}'_1(h) &= \tilde{B}_1 h^{-1/3}(1 + O(h^{-1})) + \tilde{B}_2 h^{1/3}(\tilde{a}h^{-1} + O(h^{-2})), \\ \tilde{Z}'(h) &= \tilde{B}_2 h^{1/3}(1 + O(h^{-1})) + \tilde{B}_1 h^{-1/3}((A^2 + 6A - 1)/36 + O(h^{-1})),\end{aligned}\quad (4.6)$$

where  $\tilde{B}_1, \tilde{B}_2, \tilde{a}$  are complex constants and the terms in brackets are analytic expansions of  $h^{-1}$ . This yields that along the circle  $|h| = d_\infty$ , the argument of  $\tilde{J}'_1(h)$  increase by no more than  $-2\pi/3$  as  $d_\infty \rightarrow +\infty$ . By (2.22), the change of argument of  $\tilde{J}'_1(h)$  along the circle  $|h| = d_2$  is close to zero as  $h \rightarrow h_2$ . At the end, we have shown that  $\text{Im } \tilde{J}'_1(h) \neq 0$  for  $h \in S_\pm$ , which means that along the open cut  $S_+$  (respectively  $S_-$ ), the argument of  $\tilde{J}'_1(h)$  by no more than  $\pi$ . Putting these data together yields that the increment of argument of  $\tilde{J}'_1(h)$  along  $\partial\mathcal{D}'$  is less than  $2\pi - 2\pi/3 = 4\pi/3$  as  $d_\infty \rightarrow +\infty$ ,  $d_2 \rightarrow 0$ . Using the argument principle, we obtain that  $\tilde{J}'_1(h) \neq 0$  in  $\mathcal{D}'$ . The same is true for  $\mathcal{D}$ .  $\square$

By Lemma 4.4, we can extend the definition of  $\tilde{\omega}(h)$ , defined in (4.4), from  $\Sigma = (h_1, h_2)$  to  $\mathbb{C}$ . And it is obvious that  $\tilde{\omega}(h)$  is a single valued analytic function in  $\mathcal{D}$ .

**Proposition 4.5.**  $\phi(h) \neq 0$  in  $\Sigma = (h_1, h_2)$ .

**Proof.** Let  $\tilde{\phi}(h)$  be the analytic continuation of  $\phi(h)$  from  $\Sigma = (h_1, h_2)$  to  $\mathbb{C}$ .  $\tilde{\phi}(h)$  is a single valued analytic function in  $\mathcal{D}$ .

In the proof of Lemma 4.3, we have shown that  $\lim_{h \rightarrow +\infty} v(h) = +\infty$ , which implies  $\tilde{B}_2 \neq 0$  in (4.6), otherwise  $\lim_{h \rightarrow +\infty} v(h) = \omega_\infty$ . If  $\tilde{B}_1 = \tilde{a} = 0$  in (4.6), then it follow from (4.4) and Lemma 4.3 that  $0 < A(A+3)/9 < v(h) \sim h^2 < \omega^+(h) \sim h$  as  $h \rightarrow +\infty$ , which yields contradiction. Hence,  $\tilde{a} \neq 0$  if  $\tilde{B}_1 = 0$ . By above discussions, we have either  $\tilde{\omega}(h) \sim h^{2/3}$  (if  $\tilde{B}_1 \neq 0$ ) or  $\tilde{\omega}(h) \sim h$  (if  $\tilde{B}_1 = 0$ ) as  $h \rightarrow \infty$ . Finally, we get  $\tilde{\phi}(h) \sim h$  from (2.11). By (2.21), we have  $\tilde{\phi}(h) \sim h - h_2$  as  $h \rightarrow h_2$ . Lemma 4.1 shows that  $\text{Im } \tilde{\phi}(h) = \text{Im } \tilde{\omega}(h) \neq 0$  for  $h \in S_\pm$ . Using the same arguments as in the proof of Lemma 4.4, we conclude that the increment of the argument of  $\tilde{\phi}(h)$  along  $\partial\mathcal{D}'$  is close or less than  $-2\pi + 2\pi + \pi + \pi = 2\pi$ , which implies that  $\tilde{\phi}(h)$  has at most one zero in  $\mathcal{D}'$ . On the other hand, we have shown in Lemma 2.6 that  $\phi(h_1) = 0$ . The proof is finished.  $\square$

**Proof of Theorem 1.3.** The theorem follows from Proposition 2.3, Lemmas 3.1–3.4, Proposition 4.5 and the proof for the case  $A = -2$ .  $\square$

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